Chapter 8

General Functions Method in Transport Boundary Value Problems of Elasticity Theory

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Additional information is available at the end of the chapter

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Abstract

The Lame system describing the dynamics of an isotropic elastic medium affected by a steady transport load moving at subsonic, transonic, and supersonic speed is considered. Its fundamental and generalized solutions in a moving frame of reference tied to the transport load are analyzed. Shock waves arising in the medium at supersonic speeds are studied. Conditions on the jump in the stress, displacement rate, and energy across the shock front are obtained using distribution theory. Transport boundary value problem for an elastic medium bounded by a cylindrical surface of arbitrary cross section and subjected to transport loads is considered in the subsonic and supersonic case with regard to shock waves. To solve problems, the generalized functions method is developed. In the space of generalized functions, generalized solutions are constructed and their regular integral presentations are obtained. Singular boundary equations solving the boundary value problems are presented.

Keywords: elastic medium, transport load, subsonic, transonic, supersonic speed, shock waves, boundary value problem, generalized functions method, generalized solutions, singular boundary equations

1. Introduction

A widespread source of wave generation in continuous media is transport loading, i.e., moving loads whose form does not change over time. The velocity of a transport load has a large effect on the type of differential equations describing the dynamics of the medium. The equations depend parametrically on the Mach numbers, i.e., on the ratio of the speed of motion to the propagation speeds of perturbations in the medium (sound speeds). It is well known [1] that, in an isotropic elastic medium, there are two sound speeds (c_1,c_2), which determine the velocities of dilatational and shear waves propagation. This has a large effect on the type of equations and leads to systems of elliptic, hyperbolic, or mixed equations. For transport problems, typical factors are shock effects generated by supersonic loading. At shock fronts, the stresses, displacement rates, and energy density are discontinuous. A convenient research method for such problems is provided by the theory of generalized functions (distributions), which makes it possible to significantly expand the class of processes amenable to study by using singular generalized functions in the simulation of observed phenomena. In this chapter, methods of this theory are used to solve boundary value problems using motion equations of the theory of elasticity in cylindrical domains under the action of transport loads, moving at supersonic and supersonic speeds.

2. Motion equation of elastic medium

We consider an isotropic elastic medium with Lame's parameters λ,μ, and a density ρ. Let us denote $x = x_j e_j$, e_j as the unit vectors of Cartesian coordinate system in the space R^3; displacements vector $u(\mathrm{x},t) = u_j e_j$; stress tensors σ_{ij} deformation tensor ε_{ij}. These tensors are connected by Hook's law [1]:

$$\varepsilon_{ij} = 0,5\left(u_{i,j} + u_{j,i}\right), \quad i,j,k = 1,2,3. \tag{1}$$

$$\sigma_{ij} = C_{ij}^{kl}\varepsilon_{kl} = C_{ij}^{kl}u_{k,l} \tag{2}$$

The elastic constant tensor has the symmetry properties.

$$C_{ij}^{kl} = C_{ji}^{kl} = C_{ij}^{lk} = C_{kl}^{ij}.$$

In the case of an isotropic medium, it is equal to

$$C_{ij}^{kl} = \lambda\delta_i^j\delta_l^k + \mu(\delta_i^k\delta_j^l + \delta_j^k\delta_i^l),$$

and Hook's law has the form

$$\sigma_{ij} = \lambda\,\mathrm{div}\,u\,\delta_{ij} + \mu\left(u_{i,j} + u_{j,i}\right)$$

Here $\delta_{ij} = \delta_i^j$ is the Kronecker symbol. Everywhere, there are tensor convolutions over of the same name indexes from 1 to 3, $u_{i,j} \triangleq \frac{\partial u_i}{\partial x_j}$.

Motion equations for material continuum

$$\frac{\partial\sigma_{ij}}{\partial x_j} + G_i = \rho\frac{\partial^2 u_i}{\partial t^2}, \quad i,j = 1,2,3 \tag{3}$$

for elastic medium by using Eqs. (1) and (2) have the form:

$$L_i^j(\partial_{\mathbf{x}}, \partial_t)\,u_j + G_i = 0, \tag{4}$$

Here L is the matrix Lame's operator:

$$L_i^j(\partial_{\mathbf{x}}, \partial_t) = (c_1^2 - c_2^2)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \delta_i^j\left(c_2^2\Delta - \frac{\partial^2}{\partial t^2}\right)$$

$c_1 = \sqrt{(\lambda + 2\mu)/\rho},\ c_2 = \sqrt{\mu/\rho}$ are the velocities of dilatational and shear waves ($c_1 > c_2$), $G(x,t)$ is the mass force, Δ is the Laplace operator.

The system shown in Eq. (4) was fairly well studied by Petrashen [2]. Since the elastic potential of the medium is positive definite, this system is strictly hyperbolic. Such systems can have solutions with discontinuous derivatives. The discontinuity surface F in $R^4 = R^3 \times t(-\infty < t < \infty)$ coincides with a characteristic surface of the system. It corresponds to a wave front F_t moving in R^3 at the velocity V:

$$V = -\nu_t/\|\nu\|_3, \qquad \|\nu\|_3 = \sqrt{\sum_{k=1}^{3}\nu_k^2}. \tag{5}$$

We note that $\nu(x,t) = (\nu_1,\nu_2,\nu_3,\nu_t)$ is a normal vector to F in R^4, satisfying the characteristic equation

$$\det\left\{\left(c_1^2 - c_2^2\right)\nu_i\nu_j + \delta_{ij}\left(c_2^2\|\nu\|_3^2 - \nu_t^2\right)\right\} = (c_1^2\|\nu\|_3^2 - \nu_t^2)(c_2^2\|\nu\|_3^2 - \nu_t^2)^2 = 0. \tag{6}$$

This equation has the roots:

$$\nu_t = \pm c_j\|\nu\|_3,\ \ j = 1, 2. \tag{7}$$

From Eqs. (5) and (7), we get that F_t moves in R^3 at the sound velocity $V = c_1$ or $V = c_2$.

We introduce a wave vector $m = (m_1, m_2, m_3)$. It is a unit normal vector to F_t in R^3 for fixed t in the direction of wave propagation. By virtue of Eq. (7),

$$m_j = \frac{\nu_j}{\|\nu\|_3} = -V\nu_j/\nu_t \tag{8}$$

Let $\nu_t = \nu_4$. The requirement that the displacements be continuous across the wave front, i.e.,

$$[u(x,t)]_{F_t} = 0 \tag{9}$$

which is associated with the preservation of the continuity of the medium, leads to kinematic consistency conditions for solutions at the wave front:

$$\left[m_j\frac{\partial u_i}{\partial t} + V\frac{\partial u_i}{\partial x_j}\right]_{F_t} = 0, \quad i, j = 1, 2, 3, \tag{10}$$

(the continuity of the tangent derivatives on F_t). Additionally, Eq. (4) implies dynamical consistency conditions for solutions at the wave front, which are equivalent to the momentum conservation law in its neighborhood:

$$[\sigma_{ij}]\, m_j = -\rho V \left[\frac{\partial u_i}{\partial t}\right]_{F_t}, \quad i, j = 1, 2, 3 \tag{11}$$

Definition. A wave is called a *shock wave* if the jump in the stresses across the wave front is finite: $e_i m_j [\sigma_{ij}]_{Ft} = \emptyset$. If $m_j[\sigma_{ij}]_{Ft} = 0$, then this is a *weak shock* wave. If $m_j[\sigma_{ij}]_{Ft} = \infty$, then this is a *strong shock* wave.

Velocity suffers a jump discontinuity across a shock front. At fronts of weak shock waves, the velocities are continuous, but the second derivatives of solutions are not. Strong shock waves (in the sense of the aforementioned definition) do not occur in actual media, since, at large stress jumps, the medium is destroyed and ceases to be elastic. However, strong shock waves in elastic media play an important theoretical role in the construction of solutions, specifically, fundamental solutions of Eq. (4).

3. Lame transport equations and Mach numbers

Suppose that the force affecting the medium moves at a constant velocity c along the X_3 axis (for convenience, in its negative direction) and, in a moving coordinate system $x' = (x_1, x_2, z = x_3 + ct)$ it does not depend on t:

$$G(x, z) = G_j(x_1, x_2, x_3 + ct)\, e_j \tag{12}$$

Transport solutions are solutions of Eq. (4) with the same structure:

$$\mathrm{u} = \mathrm{u}(x_1, x_2, x_3 + ct) = \mathrm{u}(\mathrm{x}, \mathrm{z}) \tag{13}$$

The speed of transport loads is called *subsonic* if $c < c_2$,*transonic* if $c_2 < c < c_1$, and *supersonic* if $c > c_1$. A speed is called the *first* or *second sound* speed if $c = c_j$, $j = 1, 2$, respectively.

In the new variables, the equations of motion are brought to the form

$$L_j^i\left(\frac{\partial}{\partial x'}\right) u_i = \left\{\left(M_1^{-2} - M_2^{-2}\right) \frac{\partial^2}{\partial x_i' \partial' x_j} + \left(M_2^{-2}\Delta - \frac{\partial^2}{\partial x_3'^2}\right) \delta_j^i\right\} u_i + g_j = 0 \tag{14}$$

Here $g_j = \left(\rho c^2\right)^{-1} G_j$; $\quad M_j = c/c_j$ are Mach numbers: ($M_1 < M_2$).

As $M_j < 1 (j = 1, 2)$ the load is subsonic and the system of equations is elliptic. If the load is supersonic, i.e., $M_j > l$, $j = 1, 2$, then the system becomes hyperbolic. In the case of transonic speeds, i.e., $M_1 < 1$ and $M_2 > 1$, the equations are hyperbolic-elliptic. In the case of sound speeds, the equations are parabolic-elliptic if $M_2 = 1$ and parabolic-hyperbolic if $M_1 = 1$. We will show this later when considering fundamental solutions of Eq. (14).

Since the original system is hyperbolic, Eq. (14) can also have discontinuous solutions. Let F be a discontinuity surface in the space of variables x' such that it is stationary in this space and moves at one of the sound velocities $V = c_1, c_2$ in the space of (x_1, x_2, x_3). It follows from Eq. (7) that $V = cn_3$, where $n = (n_1, n_2, n_3)$ is the unit normal to F in R^3. Therefore, since $c = c_j/n_3$ and $|n_3| \le 1$, such surfaces can arise only at supersonic speeds: $c \ge c_j$.

It follows from Eqs. (9) to (11) and Eq. (13) that the kinematic and dynamical consistency conditions for solutions at discontinuities in the mobile coordinate system have the form:

$$[u(x,z)]_F = 0 \quad \Rightarrow \qquad \left[n_z u_{i,j} - n_j u_{i,z}\right]_F = 0; \tag{15}$$

$$\left[\sigma_{ij}\right] n_j = -\rho c_k c [u_{i,z}]_F; \quad n_z = -c_k/c, \quad \text{for } c \ge c_k; \tag{16}$$

$n = \{n_1, n_2, n_z = n_3\}$ is a *wave vector*, $k = 1$ for shock dilatational waves, $k = 2$ for shock shear waves. Here and hereafter, the derivative with respect to x_j is denoted by the index j after a comma in the function notation or by the variable itself.

Definition. If $c > c_2$, the solution of the system in Eq. (14) is called *classical* if it is continuous and twice differentiable everywhere, except for, possibly, wave fronts. The number of fronts is finite at any fixed t and the conditions on the gaps, Eqs. (15) and (16), are satisfied on the wave fronts.

At first, we construct the solutions of the transport Lame equation using methods of generalized functions theory.

4. Shock waves as generalized solutions of transport Lame equations: conditions on wave front

Consider Eq. (14) and its solutions on the space of generalized vector functions $D'_3\,(R^3)$ with components being generalized functions from $D'(R^3)$ (see [3]). Obviously, if u is a solution of Eq. (14) that is twice differentiable, then it is also a generalized solution of Eq. (14). If a vector function u satisfies Eq. (14) in the classical sense almost everywhere, except for some surfaces, on which its derivatives are discontinuous, then, generally speaking, u is not a generalized solution of Eq. (14).

Let $u(x,z)$ be a shock wave ($x = (x_1, x_2)$), i.e., a classical solution of the Lame transport equations, Eq. (14), that satisfies conditions Eqs. (15) and (16) at the front F. Let $\widehat{u}(x,z)$ denote the corresponding regular generalized function.

Theorem 4.1. *The shock wave $\widehat{u}(x,z)$ is a generalized solution of the Lame equation in $D'_3\,(R^3)$.*

Proof. Using the rules for differentiating generalized functions with derivatives having jump discontinuities across some surfaces (see [3]), for the equations of motion in $D'_3\,(R^3)$, we obtain

$$\frac{\partial \widehat{\sigma}_{ij}}{\partial x_j'} - \rho c^2 \frac{\partial^2 \widehat{u}_i}{\partial x_z^2} + G_i = \left[\ \sigma_{ij} h_j - \rho c^2 \ h_z \frac{\partial u_i}{\partial z} \right]_F \delta_F +$$
$$+ \frac{\partial}{\partial x_j'} \left\{ \left[\lambda \ u_k h_k \delta_{ij} + \mu (u_i h_j + u_j h_i \right]_F \delta_F \right\} - \frac{\partial}{\partial z} \left\{ [u_i \ ]_F h_z \delta_F \right\}, \tag{17}$$

Here, the right-hand side involves singular generalized functions, namely, single layers $\delta_F (x,z)$ and double layers on F. By virtue of conditions Eqs. (15) and (16), the densities of these layers are equal to zero, so the right-hand side of Eq. (17) vanishes; i.e., the shock wave satisfies the same equations, Eq. (14), but in the generalized sense.

As a result, we obtain a simple formal method for deriving conditions at jumps in solutions and their derivatives across the shock fronts in hyperbolic equations. Namely, these equations are written in the space of generalized functions and the densities of the singular functions corresponding to single, double, etc., layers are set to zero.

Define as follows the *kinetic energy density*

$$\mathrm{K} = 0.5\rho \|u_{,t}\|^2 = 0.5\rho c^2 \|u_{,z}\|^2 \tag{18}$$

and *elastic potential*

$$W = 0.5\sigma_{ij} u_{i,j} = 0.5\sigma_{ij}\varepsilon_{ij} \tag{19}$$

Consider the following functions: the *energy density* $E = K + W$ of elastic deformations and the *Lagrangian* $\Lambda = K - W$.

Theorem 4.2. *If G is continuous, then the Lagrangian Λ is continuous at the shock waves fronts.*

$([\Lambda]_F = 0)$ *and the jump in the energy density satisfies the relation*

$$h_z [E]_F = \left[(\sigma_{ij} h_j) u_{i,z} \right]_F, \tag{20}$$

First formula is equivalent to the equality:

$$[E]_{F_{c_k}} = -\frac{c_k}{c} h_j^k \left[\ \sigma_{ij} u_{i,z} \right]_{F_{c_k}}, \quad k = 1, 2$$

where c_k is the sound velocity corresponding to front F, h_j^k is the components of the wave vector to F.

The last formula may be easy to get if we write the equation for E in $D_3'(R^3)$ in the form

$$\widehat{E}_{,z} = E_{,z} + [E]_F h_z \delta_F = \left(\sigma_{ij} u_{i,z}\right)_{,j} + \rho(G, u_{,z}) + \left[\sigma_{ij} u_{i,z}\right]_F \delta_F h_j + \rho\left(G, [u]_F\right) h_z \delta_F \quad \Rightarrow$$
$$[E]_F h_z = \left[\sigma_{ij} u_{i,z}\right]_F h_j$$

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as $[u]_F = 0$. For the gaps of these functions, the theorem has been proved on the basis of classic methods (see [4, 5]). For full proof of this theorem, see [6].

5. Fundamental Green's tensors and generalized solutions of transport Lame equations

The matrix of fundamental solutions $\widehat{U}(x, z)$ satisfies Eq. (14) with a delta function in the mass force:

$$L_i^j\left(\frac{\partial}{\partial \mathbf{x}'}\right)\hat{U}_j^k + \delta(\mathbf{x}')\delta_i^k = 0, \qquad i, j = 1, 2, 3. \tag{21}$$

This matrix is called *Green's tensor* for the transport Lame equations if it satisfies the decay conditions at infinity

$$\hat{U}_i^k \to 0, \quad \partial_j \hat{U}_i^k \to 0, \quad x' \to \infty, \quad i,j,k = 1,2,3. \tag{22}$$

For a fixed k, its components describe the displacements of the elastic medium under a concentrated force moving at the velocity c along the axis $Z = X_3$ and acting in the X_k direction.

Green's tensor can be obtained by taking the Fourier transform of Eq. (17) and solving the corresponding system of linear algebraic equations for the Fourier transforms $\overline{U}(\xi_1, \xi_2, \xi_3)$. It is reduced to the form (see [4]).

$$\bar{U}_i^j = \frac{M_2^2\delta_i^j}{\left(\|\xi\|^2 - M_2^2\xi_3^2\right)} + \frac{\xi_i\xi_j}{\xi_3^2}\left(\frac{1}{\|\xi\|^2 - M_2^2\xi_3^2} - \frac{1}{\|\xi\|^2 - M_1^2\xi_3^2},\right) \tag{23}$$

It can be seen that $\widehat{U}(x, z)$ has no classical inverse Fourier transform since it has non-integrable singularities in its denominators. This is associated with the fact that the matrix of fundamental solutions is defined, generally speaking, up to solutions of the homogeneous system of equations. The functions

$$\bar{f}_{km} = \xi^{-m}\left(\|\xi\|^2 - M_m^2\xi_3^2\right)^{-1}, \quad m = 0, 1, 2,$$

are of crucial importance in the construction of the original Green's tensor. It is easy to see that $\bar{f}_{0m}$ is the Fourier transform of the fundamental solution to the equation

$$\frac{\partial^2 \hat{f}_{0k}}{\partial x_1^2} + \frac{\partial^2 \hat{f}_{0k}}{\partial x_2^2} + (1 - M_k^2)\frac{\partial^2 \hat{f}_{0k}}{\partial z^2} + \delta(x)\delta(z) = 0 \tag{24}$$

This equation is similar to the elliptic Laplace equation at subsonic speeds if $M_k < 1$ and to the wave equation at supersonic speeds if $M_k > 1$. At the sound speed ($M_k = 1$), the variable z

disappears from the equation and the equation becomes parabolic, since the space dimension is higher by one, which determines the type of Eq. (14), as noted earlier, since the solutions contain waves of two types. Green's tensor for the Lame transport equation was constructed by Alekseyeva [4] by applying fundamental solutions of the Laplace and wave equations and regularization functions $\bar{f}_{km}$, which depends on the speed of transport load. Green's tensor has the regular form:

$$U_i^j(x,z) = c_2^{-2}\delta_i^j f_{02}(\|x\|,z) + c^{-2}\left(f_{21',ij}(\|x\|,z) - f_{22',ij}(\|x\|,z)\right), \tag{25}$$

where the type of basic function depends on velocity c.

In subsonic case ($M_k < 1$):

$$4\pi f_{oj}(r,z) = \frac{1}{\sqrt{z^2 + m_j^2 r^2}}, \quad 4\pi f_{1j} = \text{sgn } |z| \ \ln\left(\frac{|z| + \sqrt{z^2 + m_j^2 r^2}}{m_j r}\right),$$

$$4\pi f_{2j} = |z| \ \ln\left(\frac{|z| + V_j}{m_j r}\right) - V_j + m_j\|x\|,$$

In sonic case ($M_k = 1$):

$$f_{ok}(|x|,z) = -0.5 \ \delta(z)|x|, \quad f_{1k} = 0.5 \ \theta(z)|x|, \quad f_{2k} = 0.5 \ z\theta(z)|x|.$$

In supersonic case ($M_k > 1$):

$$f_{oj}(r,z) = \frac{\theta(z - m_j r)}{2\pi\sqrt{z^2 - m_j^2 r^2}}, \quad f_{1j} = \frac{\theta(z - m_j r)}{2\pi} \ \ln\left(\frac{z + V_j^-}{m_j r}\right), \quad f_{2j} = \frac{\theta(z - m_j r)}{2\pi}\left(z\ln\left(\frac{z + V_j^-}{m_j r}\right) - V_j^-\right),$$

Here and hereafter, we use the following notation: $\theta(z)$ is the Heaviside step function,

$$m_k = \sqrt{1 - M_k^2}, \quad V_k = \sqrt{z^2 + m_k^2 r^2}, \quad V_k^- = \sqrt{z^2 - m_k^2 r^2}, \quad r = \sqrt{x_1^2 + x_2^2} = \|x\|,$$

The dilatational and shear components of $\widehat{U}(x,z)$ are easy to write out

$$\begin{aligned} &U_i^j(x,z) = U_{i1}^j(x,z) + U_{i2}^j(x,z) \\ &U_{i1}^j = c^{-2} f_{21',ij}(\|x\|,z), \quad U_{i2}^j(x,z) = c_2^{-2}\delta_i^j f_{02}(\|x\|,z) - f_{22',ij}(\|x\|,z) \end{aligned} \tag{26}$$

In the supersonic case, the support of the functions is the cone $z > m_k\|x\|$. This determines a radiation condition as physical considerations imply that there are no displacements of the elastic medium outside this cone since the perturbations have a finite propagation velocity, which cannot be higher than the corresponding sound velocity for a particular type of deformation. At the fronts of shock waves ($z = m_k\|x\|$), Green's tensor grows to infinity.

If the following convolution exists,

$$\widehat{u}_i = \widehat{U}_i^j * G_j(x,z)/\rho c^2 \tag{27}$$

it is easy to prove that it is the generalized solution of the transport Lame equations, Eq. (14).

If mass forces are regular, then Eq. (28) has an integral presentation:

$$u_i(x,z) = \int_{D^-} U_i^j(x-y, z-\tau) g_j(y,\tau) dy_1 dy_2 d\tau = u_i(x,z) \tag{28}$$

If mass forces are concentrated on surface D and described by singular generalized functions of the type of single layers $g = g_j(y,\tau) e_j \delta_D(y,\tau)$, then

$$\widehat{u}_i(x,z) = \int_D \left(U_i^j(x-y, z-\tau) g_j(y,\tau) dD(y,\tau) = u_i(x,z) \right. \tag{29}$$

Moreover, by the Du Bois-Reymond lemma [3], these solutions are classical. For other types of singular mass forces, to calculate Eq. (28), we use the definition of convolution of a generalized function [3].

It is easy to see from Eqs. (23) to (25) that the solution is represented as a composition of fundamental solutions distributed over the support of the function *f(x,z)*; their intensities are determined by its value.

In Alexeyeva and Kayshibayeva's paper [5], there are some numerical examples of calculation of the dynamic of elastic medium at subsonic, transonic, and supersonic speed of transport loads moving along the strip in an elastic medium.

6. Subsonic Green's tensor, fundamental stress tensors, and their properties

In the subsonic case from Eq. (25), we obtain the components of Green's tensor in the form:

$$\hat{U}_1^1 = \frac{1}{4\pi c^2}\left(\frac{1}{V_2} + \frac{z^2 x_1^2 W_{12}}{r^4 M_2^2} - \frac{x_2^2 V_{12}}{r^4 M_2^2}\right),$$
$$\hat{U}_2^2 = \frac{1}{4\pi c^2}\left(\frac{1}{V_2} + \frac{z^2 x_2^2 W_{12}}{r^4 M_2^2} - \frac{x_1^2 V_{12}}{r^4 M_2^2}\right),$$
$$\hat{U}_1^2 = \hat{U}_2^1 = \frac{x_1 x_2}{4\pi c^2 r^4}\left(z^2 W_{12} + V_{12}\right),\ \hat{U}_3^3 = \frac{1}{4\pi c^2}\left(\frac{1}{V_1} - \frac{m_2^2}{V_2}\right),$$
$$\hat{U}_1^3 = \hat{U}_3^1 = -\frac{x_1 z}{4\pi c^2 r^2} W_{12}, \quad \hat{U}_2^3 = \hat{U}_3^2 = -\frac{x_2 z W_{12}}{4\pi c^2 r^2},$$
$$V_{12} = V_1 - V_2, \quad V_i = \sqrt{z^2 + m_i^2 r^2}, \quad m_i^2 = 1 - M_i^2, \quad W_{12} = V_1^{-1} - V_2^{-1},\ r = \sqrt{x_1^2 + x_2^2}$$

They are regular functions. Since by $x^{'} \to 0$ [6]:

$$V_{12} \sim \frac{r^2(m_1^2 - m_2^2)}{2\,|z|}, \quad ,W_{12} \sim \frac{r^2(m_2^2 - m_1^2)}{2\,|z|}, \quad \frac{z^2 x_1^2}{r^4} W_{12} - \frac{x_2^2}{r^4} V_{12} \sim \frac{(m_2^2 - m_1^2)}{2\,|z|} \tag{30}$$

these components are bounded for $(x, z) \neq (0, 0, 0)$. At the point $(x, z) = (0, 0, 0)$, they have a weak singularity of order R^{-1}, $R = \sqrt{z^2 + r^2}$. It has a similar asymptotic at infinity. Accordingly, R^{-2} is the order of the tensor derivatives asymptotic and the behavior of at ∞.

Tensor $\widehat{U}$ generates next fundamental stress tensors if we use Hook's law (Eq. (2)):

$$\begin{gathered} \Sigma_{jk}^i(x,z) = \lambda\, U_{l',l}^i \delta_{jk} + \mu\left(U_{j',k}^i + U_{k',j}^i\right), \quad \Gamma_j^i(x,z,n) = \Sigma_{jk}^i(x,z) n_k \\ \widehat{T}_j^i(x,z,n) = -\left(\rho c^2\right)^{-1} \Gamma_i^j(x,z,n) \end{gathered} \tag{31}$$

Then the elastic constant tensor is presented in the form

$$\widehat{T}_i^j(x,z,n) = \tilde{C}_{km}^{jl} \widehat{U}_{i',m}^k n_l, \quad \tilde{C}_{km}^{jl} = C_{km}^{jl} / \left(\rho c^2\right) \tag{32}$$

Tensor $\Gamma_j^i(x,z,n)$ describes the stresses at the plate with normal n in a point $x^{'} = (x,z)$. Tensor $\widehat{T}$ have some remarkable properties.

Theorem 6.1. *Fundamental stress tensor $\widehat{T}$ is the generalized solution of the transport Lame equation with singular mass forces of the multipole type:*

$$\rho c^2 L_i^j(\partial_{x'}) \widehat{T}_j^k + K_k^i(\partial_{x'}, n) \delta\left(x^{'}\right) = 0 \tag{33}$$

where

$$K_i^l(\partial_{x'}, n) = \lambda n_i \partial_l + \mu m_j\left(\delta_i^l \partial_j + \delta_j^l \partial_i\right).$$

For any closed Lyapunov's surface D, bounding a domain $D^- \subset R^3$

$$\delta_i^j H_D^-(x,z) = V.P. \int_D \left(T_i^j(x-y, \tau - z, n(y,\tau)) + U_{i',z}^j(x-y, \tau - z) n_z(y,\tau)\right) dS(y,\tau) \tag{34}$$

where $H_D^-(x,z)$ is the characteristic function of D^-, which is equal to 0.5 at D; $n(y,\tau)$ is a unit normal vector to D. The integrals are regular for $(x,z) \notin D$ and are taken in value principle sense for $(x,z) \in D$.

These formulas have been proved by Alexeyeva [6]. The formula in Eq. (35) can be referred to as a dynamic analog of the well-known Gauss formula for a double-layer potential of the fundamental solution of Laplace's equation ([3]: 403). It plays a fundamental role in the solution of transport boundary value problems (BVP).

7. Statement of subsonic transport boundary value problems. Uniqueness of solution

Let D^- be an elastic medium bounded by a cylindrical surface D with generator parallel to the axis X_3; let S^- be the cross-section of the cylindrical domain; let S be its boundary, and let n be the unit outward normal of D. Obviously, $n = n(x)$ and $n_3 = 0$. We assume that G is an integrable vector function and $\exists\varepsilon > 0$ such that

$$\|G(x,z)\| \le O\left(\|x'\|^{-(3+\varepsilon)}\right) \; for \|x'\| \to \infty, \quad x' \in D^- + D. \tag{35}$$

There is the subsonic transport load $P(x,z)$ moving along the boundary D ($c < c_2$):

$$\sigma_{ij}(x,z)n_j(x) = P(x,z) = \rho c^2 p_i(x,z), \quad (x,z) \in D \tag{36}$$

We assume that $\exists\varepsilon_i > 0$:

$$\|u^D(x,z)\| \le O(|z|^{-\varepsilon_1}) \quad \text{for } |z| \to \infty, \; x \in S, \tag{37}$$

$$\|p(x,z)\| \le O\left(|z|^{-1-\varepsilon 2}\right) \text{ for } |z| \to \infty, x \in S. \tag{38}$$

A vector function $u(x,z)$ satisfying the aforementioned conditions is referred to as a classical solution of the BVP. Let $C_{ab}{}^- = \{(x,z)\colon x \in D^-, a < z < b\}$. The two useful energetic equalities have been proved by Alexeyeva [6].

Theorem 7.1. *Classic solution of transport BVP satisfying to the equalities:*

$$\int_{D_{ab}} (P,u)dD(x,z) - \int_{D_{ab}^-} \left(W - 0.5\rho c^2 \|u_{,z}\|^2 - (G,u)\right) \; dx_1dx_2dz +$$

$$+ \int_{S^-} \left\{ \left(\rho c^2 u_{i,z} - \sigma_{i3}\right)\big|_{(x,a)} u_i(x,a) - \left(\rho c^2 u_{i,z} - \sigma_{i3}(x,b)\right)\big|_{(x,b)} u_i(x,b) \right\} dx_1dx_2dz = 0$$

$$\int_{S^-} \left(W + 0,5\rho c^2 \|u_{,z}\|^2 - \sigma_{i3}u_{i,z}\right)\Bigg|_z^{\pm\infty} dx_1dx_2 = \int_{D_{z,\pm\infty}} (P, u_{i,z})dx_1dx_2dz + \int_{D^-_{z,\pm\infty}} (G, u_{,z})dx_1dx_2dz$$

$$\int_D (P(x,z), u(x,z))dD(x,z) = \int_{D^-} \left(0.5\rho c^2 \|u_{,z}\|^2 - W - (G,u)\right)dx_1dx_2dz$$

$$\int_{S^-} \left(W + 0,5\rho c^2 \|u_{,z}\|^2\right)dV(x) = \int_D (P, u_{,z})dD(x,z) + \int_{D^-} (G, u_{,z})dV(x,z) \tag{39}$$

$$D_{ab} = \{(x,z) : x \in D, \; a \le z \le b\}, D_{ab}^- = \{(x,z) : x \in D^-, \; a < z < b\}.$$

The following assertion is its corollary.

Theorem 7.2. *The solution of the subsonic transport boundary value problem is unique.*

Proof. Since the problem is linear, it suffices to prove the uniqueness of the zero solution. Let $u(x,z)$ satisfy the zero boundary conditions $P(x,z) = 0$ on D and be a solution of the homogeneous Lame equations ((Eq. (14)) by $G(x,z) = 0$.

Then for $\forall z$

$$\int_{S^-} \left(W + \rho c^2 \|u,_z\|^2\right) dV(x) = 0 \tag{40}$$

It follows from the formula (Eq. (40)) of Theorem 7.1. The integrand is a positive quadratic form in $u_{i,j}$, since the elastic potential satisfies the relation $W \geq 0$ ([1]: 589, 591); moreover, $W = 0$ only for displacements of the medium treated as an absolutely rigid body. Therefore, Eq. (40) is true only if $u_{i,j} = 0$ for all i, j. This, together with the decay of solutions at infinity and the arbitrary choice of z, implies that $u = 0$.

The proof of the theorem is complete. It is valid both for the internal and external boundary value problem. The asymptotic requirements on G and the boundary functions may be weakened.

8. General functions method: statement of subsonic transport BVP in $D_3'(R^3)$

Our aim is to construct the solution of BVP by using boundary integral equations (BIE) for $u(x,z)$. The construction of an analog of Green's formula for solutions of elliptic equations ([3]: 366), which permits one to determine the values of the desired function inside the domain on the basis of the boundary values of the function and its normal derivative, is the key point in the construction of BIE of boundary value problems. An analog of this formula for equations of the static theory of elasticity is referred to as the *Somigliana formula* [1]. It determines the function $u(x,z)$ in the domain D^-, if the boundary values of displacements $u_D(x,z)$ and stresses $p(x,z)$ are given. We construct a dynamic analog of that formula in the case of transport solutions. To this end, we use the method of generalized functions (GFM).

We introduce the regular generalized solution of BVP

$$\widehat{u}(x,z) = u(x,z)H_D^-(x) = u(x,z)H_S^-(x)1(z), \tag{41}$$

which defines it as a regular vector function on all space R^3. Here $H_D^-(x,z)$ is the characteristic function of the set D: $1(z) \equiv 1$, $H_S^-(x)$ is the characteristic function of S^-, which is equal to 0.5 at S: $\partial_j H_S{}^-(x) = -n_j(x)\delta_S(x)$, where $n_j(x)\delta_S(x)$ is a simple layer at S.

By using the properties of the differentiation of regular generalized functions with jumps on D, we obtain the equation for $\widehat{u}(x,z)$:

$$\rho c^2 L_i^j(\partial_x, \partial_z,)\widehat{u}_j(x,z) = \widehat{G}_i +$$

$$+(\rho c^2 n_3 u_{i,z} - P_i)\delta_D + (n_3 u_i \delta_D)_{,z} - \left(C_{ij}^{kl} u_k n_l \delta_D\right)_{,j} \tag{42}$$

$\widehat{G} = GH_D^-(x,z)$, $\delta_D(x,z) = \delta_S(x)1(z)$, $1(z) \equiv 1$, is a simple layer on D. Since $n_3 = 0$ on D, it follows from the properties of the Green tensor that an analog of the Somigliana formula holds in the space of generalized functions:

$$\rho c^2 \widehat{u}_i = \widehat{U}_i^j * P_j \delta_D + \left(\left(\lambda u_k n_k \delta_l^j + \mu\left(n_j u_l + n_l u_j\right)\right)\delta_D{}^* \widehat{U}_i^l\right)_{,j} + \widehat{U}_i^j * G_j H_D^-,$$

which we write in a form more suitable for transformation as:

$$\widehat{u}_i = \widehat{U}_i^j * p_j \delta_D(x,z) + \widehat{U}_{i,m}^j{}^* \tilde{C}_{jm}^{kl} u_k n_l \delta_D(x,z) \tag{43}$$

If we write out this convolution in integral form with regard to the notation introduced here and Eqs. (1) and (2), then we obtain a formula, whose form coincides with the Somigliana formula for problems of elastostatics ([1]: 605):

$$u_i H_D^-(x,z) = \int_D \left(U_i^j(x,y,z,\tau)\, p_j(y,\tau) - T_i^j(x,y,z,\tau,n(y,\tau))u_j(y,\tau)\right) dD(y,\tau), \quad i,j = 1,2,3 \tag{44}$$

where we introduce the shift tensors:

$$U_i^j(x,y,z,\tau) = U_i^j(x-y, z-\tau), \qquad T_i^j(x,y,z,\tau,n) = T_i^j(x-y, z-\tau, n).$$

This formula permits one to determine displacements in the medium on the basis of known boundary values of displacements and stresses. But the integrals are regular only *for* $(x,z) \notin D$ and do not exist for $(x,z) \in D$.

9. Singular boundary integral equations of subsonic transport BVP

The following assertion provides a solution for the aforementioned boundary value problems.

Theorem 9.1. *If the solution u(x;z) of subsonic transport BVP satisfies the Holder condition on D; namely,*

$$\|u_j(x,z) - u_j(y,t)\| \le C\|(x,z) - (y,t)\|^\beta,\ x \in S,\ y \in S,$$

then u(x;z) satisfies the singular boundary integral equation

$$
\begin{aligned}
&0,5u_i(x,z) = \widehat{g}_j{}^* \widehat{U}_i^j(x,z) + \int_D U_i^j(x,y,z,\tau)\, p_j(y,\tau)dD(y,\tau) - \\
&-V.P. \int_D T_i^j(x,y,z,\tau,n(y,\tau))u_j(y,\tau)dD(y,\tau) - i,j = 1,2,3
\end{aligned}
\tag{45}
$$

Proof. Let consider Eq. (45) for $(x,z) \in D^-$. Let $(x^*,z^*) \in D$, $x' \to (x^*,z^*)$. Then, using Theorem 6.1, we have

$$
\begin{aligned}
&\lim_{(x,z)\to(x^*,,z^*)} u_i(x,z) = u_i(x^*,z^*) = \widehat{g}_j{}^* \widehat{U}_i^j(x^*,z^*) + \lim_{(x,z)\to(x^*,z^*)} \int_D U_i^j(x,y,z,\tau)\, p_j(y,\tau)dD(y,\tau) - \\
&- \lim_{(x,z)\to(x^*,z^*)} \int_D T_i^j(x,y,z,\tau,n(y,\tau))\big(u_j(y,\tau) - u_j(x^*,z^*)\big)dD(y,\tau) + \\
&+u_j(x^*,z^*) \lim_{(x,z)\to(x^*,z^*)} \int_D T_i^j(x,y,z,\tau,n(y,\tau))dD(y,\tau) = \\
&= \widehat{g}_j{}^* \widehat{U}_i^j(x^*,z^*) + \int_D U_i^j(x,y,z,\tau)\, p_j(y,\tau)dD(y,\tau) - \\
&-V.P. \int_D T_i^j(x,y,z,\tau,n(y,\tau))\big(u_j(y,\tau) - u_j(x^*,z^*)\big)dD(y,\tau) + \\
&+u_j(x^*,z^*) \lim_{(x,z)\to(x^*,z^*)} \left(\delta_i^j - \int_D U_{i',z}^j(x,y,z,\tau))n_z(y)dD(y,\tau) \right) = \\
&= \widehat{g}_j{}^* \widehat{U}_i^j(x^*,z^*) + \int_D U_i^j(x,y,z,\tau)\, p_j(y,\tau)dD(y,\tau) - \\
&-V.P. \int_D T_i^j(x,y,z,\tau,n(y,\tau))\big(u_j(y,\tau) - u_j(x^*,z^*)\big)dD(y,\tau) + \\
&+u_j(x^*,z^*)\delta_i^j = \widehat{g}_j{}^* \widehat{U}_i^j(x^*,z^*) + \int_D U_i^j(x,y,z,\tau)\, p_j(y,\tau)dD(y,\tau) - \\
&-V.P. \int_D T_i^j(x,y,z,\tau,n(y,\tau))u_j(y,\tau)dD(y,\tau) - 0,5u_i(x^*,z^*) + u_i(x^*,z^*).
\end{aligned}
$$

In the last relation, we have used the obvious properties: integrals with U_i^j exist by virtue of the Holder property of u on D and weak singularity U_i^j at D. Then if the surface integral exists, its value coincides with the principal value; the principal value of the integral containing the difference of integrated functions is equal to the difference of the principal values of integrals corresponding to each of these functions if they exist.

By transposing the last two terms to the left-hand side of the relation, we obtain the formula of the theorem for the boundary points. The proof of the theorem is complete.

This theorem gives us resolving system of integral equations for defining unknown values of boundary displacements.

Note also that the subsonic analog of the Somigliana formula was obtained for generalized functions. But since they are regular, from the Dubois-Reymond lemma ([3]: 97), the solution is classical. However, if the acting loads are described by singular generalized functions, which often takes place in physical problems, then one should use a representation of a generalized solution in the convolution form (Eq. (43)) with the evaluation of convolutions by the definition (see [3]: 133).

10. Supersonic green's tensor and its antiderivative with respect to *z*

From Eq. (25), we get the regular representation of $\widehat{U}_i^j$ in the supersonic case which has the form

$$
\begin{aligned}
2\pi U_1^1 &= \frac{\theta_2}{V_2} + \frac{z^2 x_1^2}{r^4 M_2^2}\left(\frac{\theta_1}{V_1} - \frac{\theta_2}{V_2}\right) - \frac{x_2^2}{r^4 M_2^2}\,(\theta_1 V_1 - \theta_2 V_2),\\
2\pi U_2^2 &= \frac{\theta_2}{V_2} + \frac{z^2 x_2^2}{r^4 M_2^2}\left(\frac{\theta_1}{V_1} - \frac{\theta_2}{V_2}\right) - \frac{x_1^2}{r^4 M_2^2}\,(\theta_1 V_1 - \theta_2 V_2),
\end{aligned}
\tag{46}
$$

$$
2\pi U_1^2 = \frac{x_1 x_2}{r^4}\left(z^2\left(\frac{\theta_1}{V_1} - \frac{\theta_2}{V_2}\right) + \ (\theta_1 V_1 - \theta_2 V_2)\right), 2\pi U_3^3 = \left(\frac{\theta_1}{V_1} + \frac{\theta_2 m_2^2}{V_2}\right),
$$

$$
2\pi U_1^3 = -\frac{x_1 z}{r^2}\left(\frac{\theta_1}{V_1} - \frac{\theta_2}{V_2}\right), 2\pi U_2^3 = -\frac{x_2 z}{r^2}\left(\frac{\theta_1}{V_1} - \frac{\theta_2}{V_2}\right)
$$

Here $\theta_j = \theta(z - m_j\|x\|), V_j^- = \sqrt{z^2 - m_j^2\|x\|^2},\ \ m_j = \sqrt{M_j{}^2 - 1}$. It satisfies the radiation conditions:

$$
\operatorname*{supp}_z\ U(x,z) \in \{z > 0\}, U_i^k \to 0, U_{i,j}^k \to 0 \text{ by } x' \to \infty. \tag{47}
$$

One can readily see that its components are zero outside the sonic cones:

$$
K_l{}^+ = \{(x,z) : z > m_l\|x\|\}, l = 1, 2.
$$

On the surfaces of the cones, the components U_1^3 have singularities of the type $(z^2 - m^2{}_j r^2)^{-1/2}$.

For solution of supersonic problems, we introduce the tensor $\widehat{W}_j^i(x,z)$, which is the antiderivative of $\widehat{U}_j^i$ with respect to z:

$$\widehat{W}_j^i = \sum_{k=1}^{2} \widehat{W}_{jk}^i = \widehat{U}_j^i * \delta(x_1)\delta(x_2)\theta(z) = \widehat{U}_j^i \underset{z}{*} \theta(z), \quad \widehat{W}_{j',z}^i = \widehat{U}_j^i \tag{48}$$

They are also fundamental solutions of Eq. (14) for the mass forces of the corresponding $F_j \underset{z}{*} \theta(z)$. After calculation, we define its components as:

$$\begin{gathered}
2\pi W_1^1 = \frac{z}{r^4}\left(x_1^2 - x_2^2\right)(\theta_1 V_1 - \theta_2 V_2) + 0,5m_1^2\theta_1 \ln\frac{z+V_1}{m_1 r} + \left(M_2^2 - 0,5m_2^2\right)\theta_2 \ln\frac{z+V_2}{m_2 r} \\
2\pi W_2^2 = -\frac{z}{r^4}\left(x_1^2 - x_2^2\right)(\theta_1 V_1 - \theta_2 V_2) + 0,5m_1^2\theta_1 \ln\frac{z+V_1}{m_1 r} + \left(M_2^2 - 0,5m_2^2\right)\theta_2 \ln\frac{z+V_2}{m_2 r}
\end{gathered} \tag{49}$$

$$2\pi W_3^3 = \theta_1 \ln\frac{z+V_1}{m_1 r} + m_2^2\theta_2 \ln\frac{z+V_2}{m_2 r}, \quad 2\pi W_2^3 = -x_2 r^{-2}(\theta_1 V_1 - \theta_2 V_2)$$

$$2\pi W_1^2 = z x_1 x_2 r^{-4}(\theta_1 V_1 - \theta_2 V_2), \quad 2\pi W_1^3 = -x_1 r^{-2}(\theta_1 V_1 - \theta_2 V_2)$$

Tensor $\widehat{W}_j^i$ has the same support as $\widehat{U}_j^i$ but as at the cone K_j

$$m_j\|x\| = z \ \Rightarrow V_j^-(x,z) = 0 \quad \Rightarrow \ln\frac{z+V_j}{m_j\|x\|} = 0 \tag{50}$$

it continues on fronts K_j. $W_j{}^i(x,z)$ has weak singularity by $x' = 0$ and weak logarithmic singularity on Z with respect to $\|x\|$ by $x = 0$. To single out these singularities, we decompose it into the terms:

$$\begin{gathered}
W_j^i(x,z) = W_j^{is}(x,z) + W_j^{id}(x,z) = \sum_{k=1}^{2}\theta_k(z - m_k r) \ \ W_{jk}^{is}(x) + W_j^{id}(x,z), \\
2\pi c^2 W_{j1}^{is}(x) = -\left(\delta_{i3}\delta_{j3} + 0,5m_1^2(1-\delta_{i3})\delta_{ij}\right)\ln m_1 r, \\
2\pi c^2 W_{j2}^{is}(x) = \left(\delta_{i3}\delta_{j3} + \delta_{ij}\left(0,5m_1^2(1-\delta_{i3}) - M_2^2\right)\ln m_2 r\right.
\end{gathered} \tag{51}$$

The tensors W_j^{is} of diagonal form are independent of z inside the sonic cones $K_l(l = 1, 2)$ and have a logarithmic singularity with respect to $\|x\|$ on the Z-axis. Unlike the generating tensor W_j^{is}, W_j^{id} has bounded jumps on the K_l. One can readily see that the tensor shifts

$$U_i^j(x,y,z) = \widehat{U}_i^j(x-y,z), W_i^j(x,y,z) = \widehat{W}_i^j(x-y,z)$$

have the following symmetry properties around the Z-axis:

$$U_i^j(x,y,z) = U_i^j(y,x,z) = U_j^i(x,y,z), \quad W_i^j(x,y,z) = W_i^j(y,x,z) = W_j^i(x,y,z), \ \ i,j = 1,2 \tag{52}$$

But for the components with indices $(i,j) = (1,3), (3,1), (2,3), (3,2)$

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$$U_i^j(x,y,z) = -U_i^j(y,x,z), \quad W_i^j(x,y,z) = -W_i^j(y,x,z) \tag{53}$$

11. Fundamental supersonic antiderivative stress tensor $\widehat{H}$ and its properties

We introduce antiderivative stress tensor

$$\begin{aligned} \tilde{\Sigma}_{i3}^j &= \widehat{\Sigma}_j^i * \theta(z)\delta(x) \\ \widehat{H}_j^i &= \widehat{T}_j^i * \theta(z)\delta(x) = \widehat{T}_j^i \underset{z}{*} \theta(z), \widehat{H}_{j,z}^i = \widehat{T}_j^i \end{aligned} \tag{54}$$

This tensor can be obtained in a different way, by analogy with T, using Hooke's law, except that the Green tensor should be replaced with its antiderivative W. By using the presentation of the basic functions of Green's tensor construction (Eq. (25)) in the supersonic case, it can be presented in the following form:

$$\hat{H}_i^j(x,z,n) = \left(2M_1^2 - M_2^2\right) n_j f_{11,i} - M_2^2 \left(\delta_{ij}\frac{\partial f_{12}}{\partial n} + n_i f_{12,j}\right) - 2\frac{\partial}{\partial n}\left(f_{31,ij} - f_{32,ij}\right)\Big)$$

$$2\pi f_{1k,i}\left(\|x\|, z\right) = \frac{\theta_k}{V_k^-}\left(\delta_{i3} - \frac{z}{\|x\|}r_{,i}\right),$$

$$\begin{aligned} 2\pi f_{3k,ij}\left(\|x\|, z\right) &= \left(\delta_{i3}\delta_{j3} + 0,5m_k^2\delta_{ij}\epsilon_{[i]3}\right)\theta_k \ln\frac{z+V_k^-}{m_k\|x\|} - \\ &- \frac{V_k^-\theta_k}{\|x\|}\left(\delta_{i3}r_{,j} + \delta_{j3}r_{,i} + \frac{z}{\|x\|}\left(r_{,i}r_{,j} - 0,5\delta_{ij}\epsilon_{[i]3}\right)\right) \end{aligned}$$

Obviously, for $z < \tau$, all the introduced shifted tensors are zero. It has the following symmetry properties around the Z-axis:

$$H_i^j(x,y,z,m) = -H_i^j(y,x,z,m) = -H_i^j(x,y,z,-m)$$

except for $(i,j) = (1,3), (2,3), (3,1)$:

$$H_i^3(x,y,z) = H_i^3(y,x,z), \quad H_i^3(x,y,z) = H_i^3(y,x,z), \quad i = 1, 2.$$

Components $H_j^i(x,z)$ have weak singularities on the fronts of the type $\left(z^2 - m^2{}_j\|x\|^2\right)^{-1/2}$, but more stronger singularity of the type of $\|x\|^{-1}$ on the axis Z. If we put Eq. (51) in Hook's law, then we can again single out two terms in $H_j^i(x,z)$:

$$H_j^i(x,z) = H_j^{is}(x,z) + H_j^{id}(x,z) = \sum_{k=1}^{2}\theta_k(z - m_k r)\left(H_{jk}^{is}(x) + H_{jk}^{id}(x,z)\right) \tag{55}$$

Since the tensors $H^{is}_{jk}(x)$ independent of z inside the sonic cones K_l (l = 1,2), we conventionally say that they are *stationary*. Accordingly, the tensors $H^{id}_{j}(x,z)$ are said to be *dynamic*, because they depend essentially on z, although they are regular functions. The aforementioned symmetry properties hold for both stationary and dynamic terms in the tensors.

For this type of tensors, the next theorem was proved (see [7]).

Theorem 11.1. *The fundamental stress tensor H satisfies the relation*

$$\delta_i^j H_S^-(x)\theta(z) = \int_0^z d\tau \int_S H_i^j(y-x,\tau,n(y))\ dS(y) +$$
$$+\int_S \left(\left(\rho c^2\right)^{-1} \tilde{\Sigma}_{i3}^j(x-y,z) - U^j_{i',z}(y-x,z) \right) dy_1 dy_2$$

For $x \notin D$ all integrals are regular; for $x \in D$ the first integral is singular, calculated in value principle sense.

This theorem enables us to obtain solvable singular boundary integral equations for a supersonic transport boundary value problem.

12. Statement of supersonic transport BVP: uniqueness of solutions

We suppose here that supersonic transport loads, moving at supersonic velocity $c > c_1$, are known on the boundary D:

$$P = \sigma_{ij} n_i e_j = \rho c^2 p_j(x,z) e_j \theta(z), x = (x_1, x_2) \in S, \qquad i,j = 1,2,3 \tag{56}$$

Functions $p_j(x,z)$ are integrable on D_+. We assume here $G = 0$ and

$$u(x,z) = 0, \quad u_{i,z}(x,z) = 0, \qquad z \le 0, \quad x \in S^- \tag{57}$$

For $\|(x,z)\| \to \infty$

$$u_j \to 0, \exists \varepsilon > 0 : \ \|\partial_j u\| < O\left(\|(x,z)\|^{1+\varepsilon}\right), \qquad j = 1,2,z \tag{58}$$

The jump conditions, Eqs. (15) and (16) are satisfied on the shock wave fronts.

Theorem 12.1. *The solution of the supersonic transport boundary value problem is unique.*

Proof. Suppose that there exist two solutions. Since the problem is linear, it follows that their difference $u(x,z)$ satisfies the zero boundary conditions, i.e., $P(x;z) = 0$, and is a solution of the homogeneous equations of motion (G = 0). We note, that Lemma 8.1 is also true in the supersonic case for shock waves as there is Theorem 3.2 for the gaps of energy on their fronts (see full proof). Then together with conditions given in Eq. (59) of decay of the solutions at infinity and the zero conditions for $z = 0$,

$$\int_{S^-} E(x,z)dx_1dx_2 = \int_{S^-} \sigma_{i3}u_{i,z}(x,z)dx_1dx_2 \to 0 \quad by \quad z \to \infty$$

The energy density E is a positive definite quadratic form of $u_{i,j}$ by construction. Therefore, by virtue of the decay of the solution at infinity, the relation only holds if $u_{i,j} = 0$ for all i and j. Hence, we obtain $u = 0$; i.e., the solutions coincide. The proof of the theorem is complete.

Theorem 12.1 holds for both exterior and interior boundary value problems.

13. Statement of supersonic BVP in $D_3'(R^3)$ and its generalized solution

To solve the problem, we also use the method of generalized functions. We introduce here the regular generalized function with support on D_+^-:

$$\widehat{u}_j(x,z) = u_j(x,z)H_S^-(x)\theta(z) \tag{59}$$

Also using the properties of differentiation of regular generalized functions with gaps at D, and taking into account the boundary conditions and the conditions on the fronts, we obtain the transport Lame equations (Eq. (14)) on the space of distributions with singular mass forces:

$$\widehat{g}_j = p_j\delta_S(x)\theta(z) + \left(\left(\lambda u_k n_k \delta_{ij} + \mu\left(u_i n_j + u_j n_i\right)\right)\delta_S(x)\theta(z)\right)_{,i} \tag{60}$$

By using the properties of convolutions with the Green tensor and the boundary conditions, we obtain the generalized solution of BVP in the form:

$$\rho c^2 \widehat{u}_k = \widehat{U}_k^j * P_j\delta_S(x)\theta(z) + \widehat{U}_{k,i}^j * \left(\lambda u_m n_m \delta_{ij} + \mu\left(u_i n_j + u_j n_i\right)\right)\delta_S(x)\theta(z) \tag{61}$$

By analog with the subsonic case, if we use fundamental stress tensor, then the right-hand side of Eq. (61) may be represented in the form of a surface integral over the boundary of the domain. In our notation, on the boundary, it acquires the form

$$u_i H_S^-(x)\theta(z) = \int_{D_+} \left(U_i^j(x,y,z-\tau)\, p_j(y,\tau) - T_i^j(x,y,z-\tau,n(y,\tau))u_j(y,\tau)\right)dD(y,\tau) \tag{62}$$

This formula is similar to the Somigliana formula in the static theory of elasticity ([1]: 146), but it is impossible to use this formula to determine the solution of the boundary value problem in the case of supersonic loads, because the second term contains strong non-integrable singularities of the tensor T on the shock wave fronts of fundamental solutions; therefore, the integrals are divergent. To construct a regular integral representation of the formula, we must regularize it. For this, we use the tensor H.

14. Dynamic analog of the Somigliana formula in supersonic case

For regularization of Eq. (61), we put $W_{,z}$ instead of U in the second term and use the property of differentiation of convolution:

$$
\begin{aligned}
&\rho c^2 \widehat{u}_k = \widehat{U}_k^j * P_j \delta_S(x)\theta(z) + \widehat{W}_{k',iz}^{j} * \left(\lambda u_m n_m \delta_{ij} + \mu\left(u_i n_j + u_j n_i\right)\right)\delta_S(x)\theta(z) = \\
&= \widehat{U}_k^j * P_j \delta_S(x)\theta(z) + \overleftarrow{W}_{k',i}^{j} * \left(\lambda u_{m,z} n_m \delta_{ij} + \mu\left(u_{i,z} n_j + u_{j,z} n_i\right)\right)\delta_S(x)\theta(z) + \\
&+ \widehat{W}_{k',i}^{j} * \left(\lambda u_m n_m \delta_{ij} + \mu\left(u_i n_j + u_j n_i\right)\right)\delta_S(x)\delta(z) = \\
&= \widehat{U}_k^j * P_j \delta_S(x)\theta(z) + \widehat{W}_{k',m}^{j} * C_{jm}^{il} u_{i,z} n_l(x)\delta_S(x)\theta(z) + C_{jm}^{il} \widehat{W}_{k',m}^{j} \underset{x}{*} u_i(x.0) n_l(x)\delta_S(x)
\end{aligned}
\tag{63}
$$

From here on, we use Eq. (57) we get the formula which can be written in integral form.

Theorem 14.1. *The generalized solution of supersonic transport BVP can be presented in the form:*

$$
\widehat{u}_k = \widehat{U}_k^j * p_j \delta_S(x)\theta(z) + \tilde{C}_{jm}^{il} \widehat{W}_{k',m}^{j} * u_{i,z} n_l(x)\delta_S(x)\theta(z) \tag{64}
$$

which for $x \notin S$ has the next integral presentation

$$
\begin{aligned}
u_i H_S^-(x)\theta(z) = &\sum_{k=1}^{2} \int_S \theta(z - m_k r) dS(y) \int_0^{z - m_k r} \left\{ U_i^j(x - y, z - \tau)\, p_j(y,\tau) - \right. \\
&\left. - H_i^{jd}(x - y, z - \tau, n(y)) u_{j,z}(y,\tau) \right\} d\tau - \int_S H_i^{js}(x - y, z, n(y)) u_j(y, z - m_k r) dS(y)
\end{aligned}
\tag{65}
$$

$$
r = \|x - y\|
$$

Proof. Formula (65) follows from Eq. (64) in virtue of Eqs. (61) and (32). Its integral form is

$$
u_i H_S^-(x)\theta(z) = \int_0^z \left\{ U_i^j(x, y, z - \tau)\, p_j(y,\tau) - H_i^j(x, y, z - \tau, n(y)) u_{j,z}(y,\tau) \right\} d\tau
$$

If we use Eq. (55) for $H_j^i(x,z)$ as the support of $H_j^{is}(x,z)$, $H_j^{id}(x,z)$, we get

$$
\begin{aligned}
u_i H_S^-(x)\theta(z) = &\sum_{k=1}^{2} \int_S \theta(z - m_k r) dS(y) \int_0^{z - m_k r} \left\{ U_i^j(x - y, z - \tau)\, p_j(y,\tau) - \right. \\
&\left. - \left(H_{ik}^{jd}(x - y, z - \tau, n(y) + H_{ik}^{js}(x - y, n(y))\right) u_{j,z}(y,\tau) \right\} d\tau
\end{aligned}
\tag{66}
$$

Note that

$$\int_0^{z-m_k r} H_{ik}^{js}(x-y,n(y))u_{j,\tau}(y,\tau)d\tau = H_{ik}^{js}(x-y,n(y)))(u_j(y,z-m_k r)-u_j(y,0) =$$
$$= H_{ik}^{js}(x-y,n(y)))u_j(y,z-m_k r)$$

In virtue of this equity, we get from Eq. (66) the last formula of the theorem.

All integrals exist; indeed, the integrands are integrable everywhere, including the fronts of fundamental solutions, because the kernels of the integrands have weak singularities on the fronts of the form $\left(z^2 - m^2{}_j\|x\|^2\right)^{-1/2}$ in virtue of the properties of kernels U and H. The proof is completed.

This formula is a dynamic analog of Somigliana formula for supersonic loads. It defines the displacement in elastic medium by using boundary values of stresses and velocity of displacements of boundary surface.

This formula also preserves its form for $(x,z)\in D$ with regard to the definition of $H_S{}^{-}(x)\theta(z)$ on D.

15. Singular boundary integral equations of supersonic transport BVP

Theorem15.1. *If the classical solution of BVP satisfies the Holder's conditions at D_+, i.e., $\exists C>0$, $\beta>0$ that*

$$\left|u_j(x,z)-u_j(y,z)\right| < C\|x-y\|^\beta, \quad x,y\in S.$$

then it satisfies the singular boundary integral equation at D_+

$$0,5u_i(x,z) = \sum_{k=1}^{2}\int_{S_z^k(x')}\theta(z-m_k r)dS(y)\int_0^{z-m_k r}\left\{U_i^j(x-y,z-\tau)\,p_j(y,\tau)-\right.$$
$$\left.-H_i^{jd}(x-y,z-\tau,n(y))u_{j,z}(y,\tau)\right\}d\tau-$$
$$-V.P.\int_{S_z^k(x)} H_i^{js}(x-y,z,n(y))u_j(y,z-m_k r)dS(y), r=\|x-y\|$$

$$S_\tau^k(x') = \{(y,\tau): m_k r < z-\tau\}, \quad S_z^k(x) = \{(y): m_k r < z\}.$$

Proof. The desired assertion follows from Theorem 14.1 and Theorem 11.1 for tensor H by analogy of the proof of Theorem 12.1 about singular boundary integral equations in the subsonic case. Full proofs of these theorems can be found in [7].

This theorem gives us a resolving system of integral equations for definition of unknown values of boundary displacements in the supersonic case.

Moreover, the Somigliana formula for displacements was obtained for generalized functions. But since they are regular, from the Dubois-Reymond lemma ([3]: 97), this solution is classical. However, if the acting loads are described by singular generalized functions, which often takes place in physical problems, then one should use a representation of a generalized solution in the convolution form (Eq. (65)) with the evaluation of convolutions by the definition (see [3]: 133).

16. Conclusion

The constructed singular boundary integral equations in the supersonic case are not classical equations because the solution inside a domain is determined by the boundary values of stresses and displacement rates rather than displacements themselves, unlike the Somigliana formula. In addition, the domain of integration over a boundary surface substantially depends on z, which is specific for hyperbolic equations. This complicates finding solutions of such problems by the successive approximation method. However, for the numerical discretization of singular boundary integral equations, the method of boundary elements makes it possible to use standard methods of computational mathematics for a computer implementation of the solution of such problems. The aforementioned boundary value problems model the dynamics of underground structures like transport tunnels and extended excavations subjected to the dynamic influence of moving vehicles and seismic loads. They permit one to study the dynamics of a rock mass in a neighborhood of underground structures depending on its physical-mechanical properties, the velocity of moving transport, specific features of the transport load, and the geometric properties of structures in technical computations of displacements and the stress-strain state of the mass away from the tunnel.

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